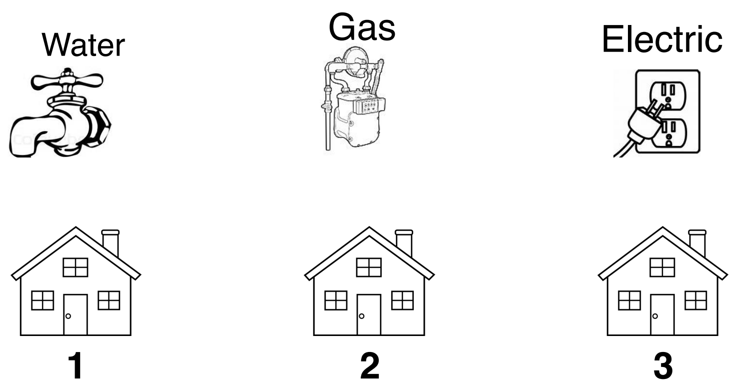
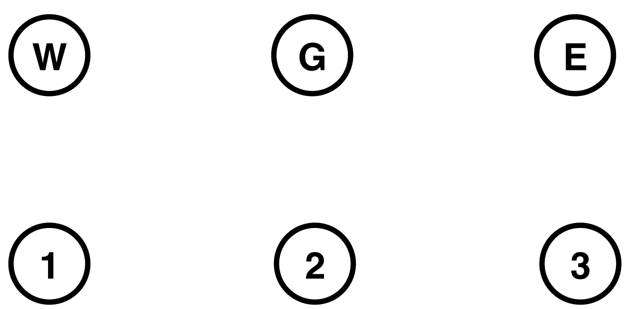
Water Gas and ElectricIty Puzzle.
The Three Cottage Problem.
The Impossible Puzzle.

Three houses all need to be supplied with water, gas and electricity. Supply lines from the water, gas and electric utilities will be used to supply the three houses. The layout is shown below.



Draw lines to get the water, gas and electric utilities into every house.   You need 3 lines from the water faucet **W** to houses **1 ,2 and 3** . You need 3 lines from the gas meter **G** to each of the 3 houses **1 ,2 and 3**.   You need 3 lines from the electric plug **E** to each of the 3 houses **1 ,2 and 3**. You must stay on the surface of the page when you draw the lines. No line can cross another line. You may want to use **3 different colors, one for each of the different types of utility lines** to make it easier to keep tract of the lines.

Use the diagram below to show the paths to connect the utility lines to each house.



Use this worksheet with 4 small puzzles if you need to make multiple attempts.

W G E

1 2 3

W G E

1 2 3

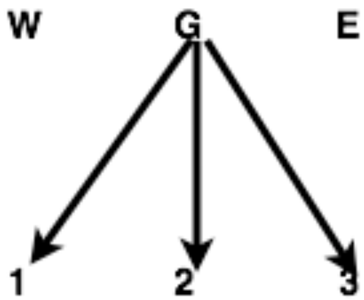
W G E

1 2 3

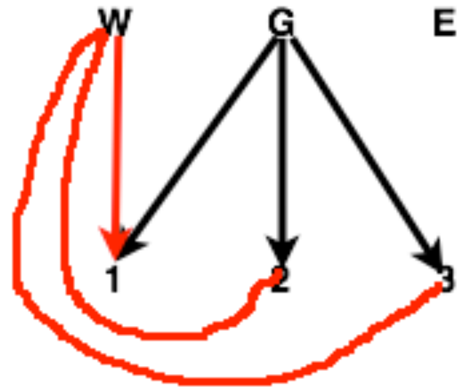
W G E

1 2 3

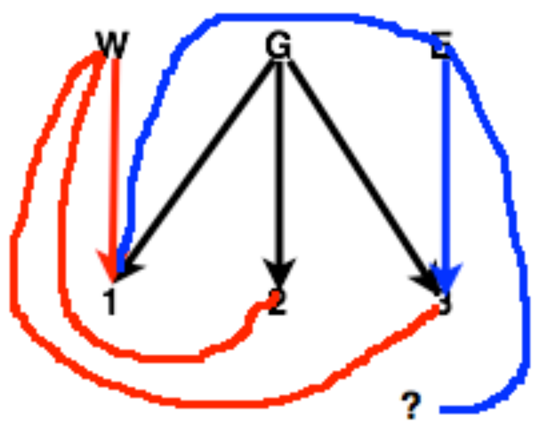
There are many ways to get started. It is easy to connect all 3 houses to any one utility. Here I started with **G**as.



To connect the next utility to the 3 houses requires curved supply lines. I used red supply lines to connect the **W**ater.



I used blue lines to try and connect the last utility **Electricity** to each the 3 houses. It was easy to connect 2 of the houses with the blue lines, but there was no way to connect house 2 without crossing the black or red lines.



This is only one of many different pathways that a student can attempt to find a solution. Many will come up to you with a drawing that they thinks work but after you help them find their mistake they will see it does not work

How can I say that there drawing cannot be correct? Because, if they follow the directions as they understand them and stay on the 2 dimensional plane of the paper **there is no solution to the problem.**

Why I suggest you use this puzzle with your students.

1. It is a classic puzzle. For over 100 years teachers have presented this puzzle to their students. After allowing the students to try as many pathways as possible the teacher is then able to help lead them towards a solution. T**his solution is based on expanding the problem into 3 dimensions.** It is a great example of "thinking outside the box"

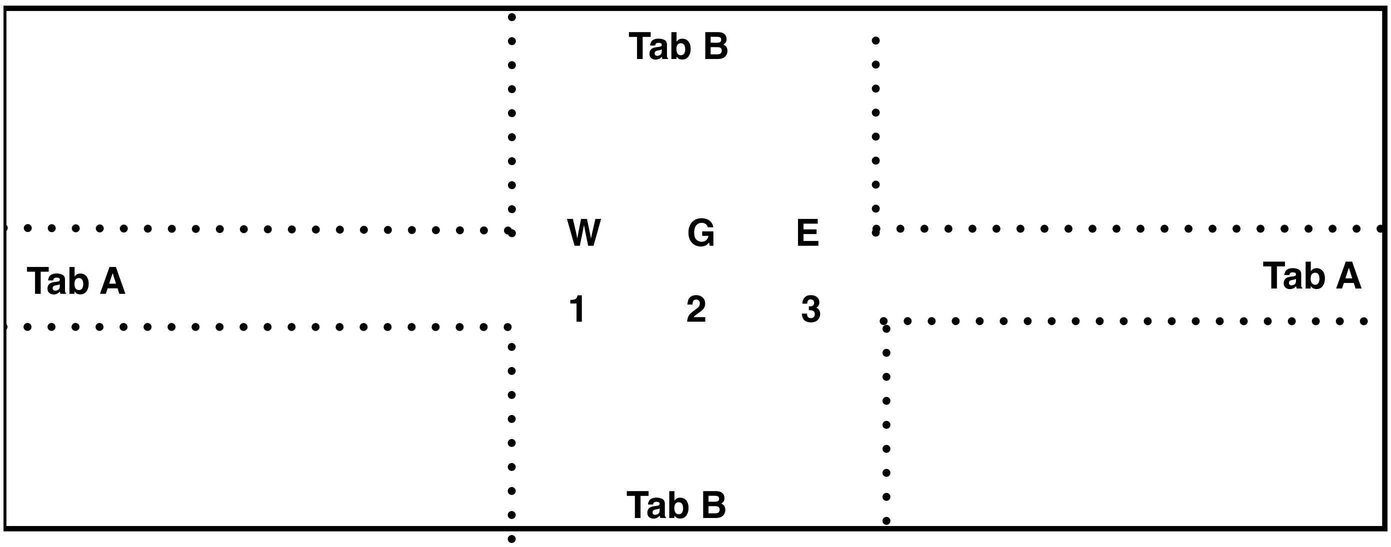
2. Network problems from a major area of study in advanced mathematics. This problem can be a culminating final problem resented to upper elementary students as a unit on f pathway and network problems. There are a few simple equations that form the basis for solving all the network problems.

3. Expanding the problem into 3 dimensions also allows the teacher to introduce simple algebraic Euler's Formula for polyhedra. This is well within the scope of an good algebra student.

Solution 1:

The problem states that "You must stay on the surface of the page when you draw the lines." It does not require the surface be flat or part of a plane. The solution below changes the flat surface of the paper into a 3 dimensional form but allows the lines to stay on a single surface.

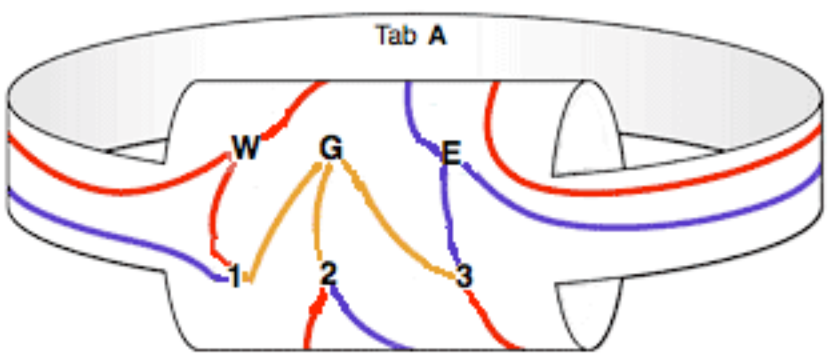
Cut out the rectangle below. Then cut out the "rectangular 4 corners" along the dotted lines. Wrap the paper lengthwise to form a tube and tape the tab B's together. Now wrap the "flaps" with Tab A's at the ends together behind the tube and tape the tab A's together. There is a single surface around the tube as well as around the thin flaps. The rectangle on the next page is a better size for drawing the lines. Use q different color for each utility to make the paths easier to follow.



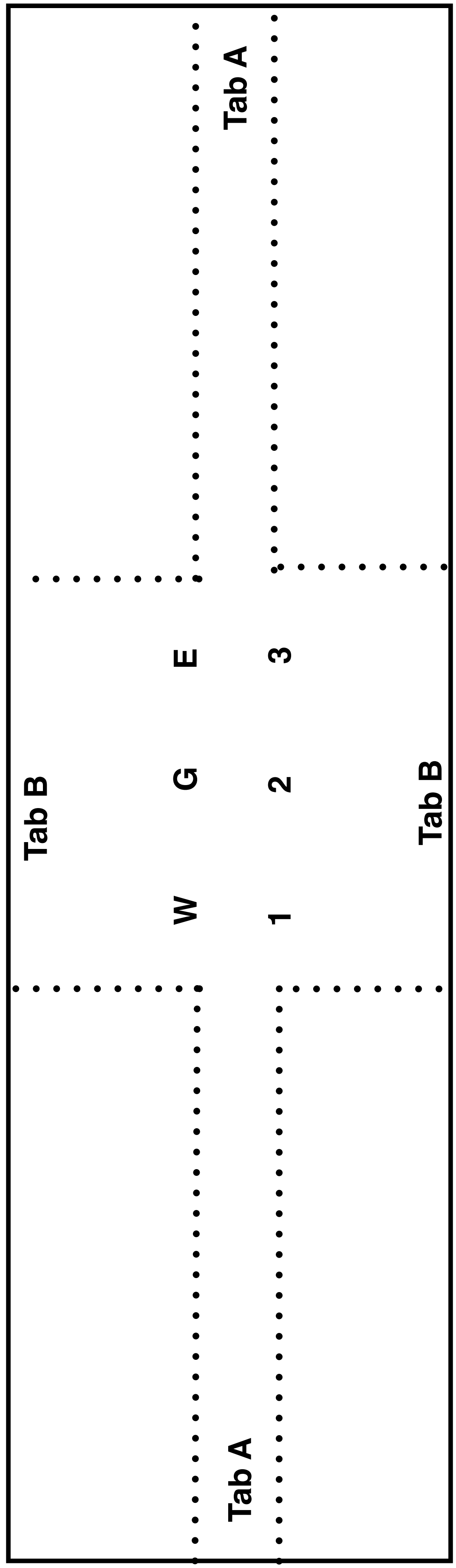
Now connect the gas to the 3 houses with the gold lines.

Then connect the electricity by connecting E to house 3 with the short red line. Next draw a purple line around the back of the tube and come out on the front and connect to house 2. Then draw a purple line from E to the right all the way around the thin flap coming back in front on the left side and connect to house 1.

Then connect the water by connecting W to house 1 with the short red line. Next draw a red line around the back of the tube and come out on the front and connect to house 2. Then draw a red line from W to the left all the way around the thin flap coming back in front on the right side and continue around the back of the tube and back to the front and connect to house 3.

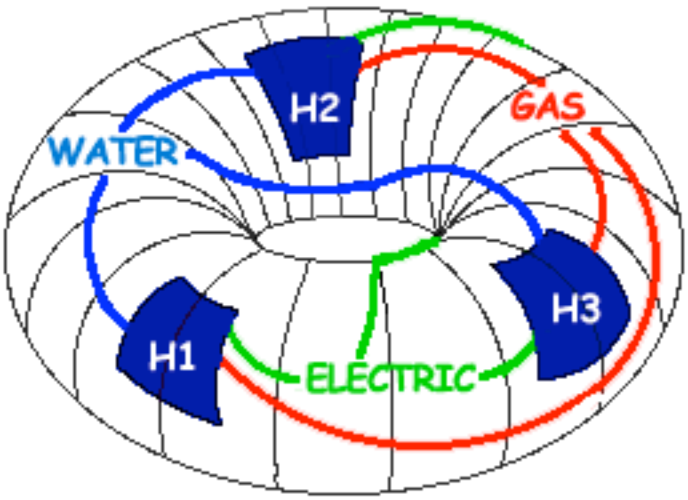


The lines all stay on the same front surface and none of the lines cross. What a surprising solution.

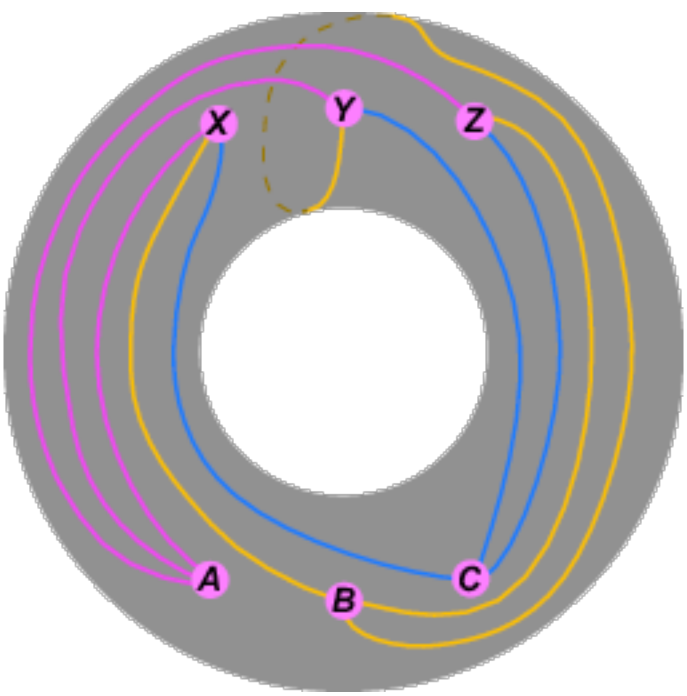

Tab A
Tab B
W G E
1 2 3
Tab B
Tab A

Solution 2 based on a different wording of the problem:

The problem states that “You must stay on the surface of the page when you draw the lines.” If it said that “ “all lines must stay on the same surface when you draw the lines”. It there is no requirement the surface of a page be used then there is another solution. Draw the 3 houses and 3 utilities on the surface of a bagel and connect them as shown below. **The surface of a bagel is a real world example of a torus**.

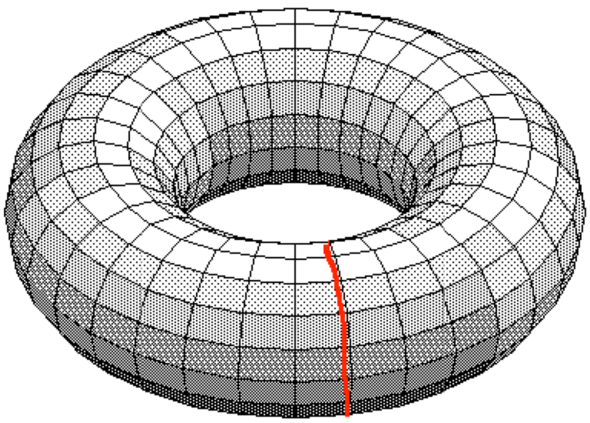


The torus below displays a different set of paths that solve the problem



A torus is the **surface** of a three-dimensional figure shaped like a doughnut. It is hollow.

If you cut the torus shown below around where the red line and could straighten it out you would form a long straight tube



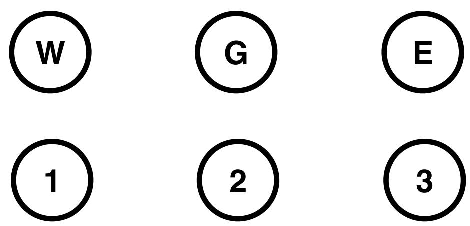
If you then cut the tube lengthwise and spread it flat you would get a flat piece of paper with 2 separate sides.

The Origin Of The Puzzle.

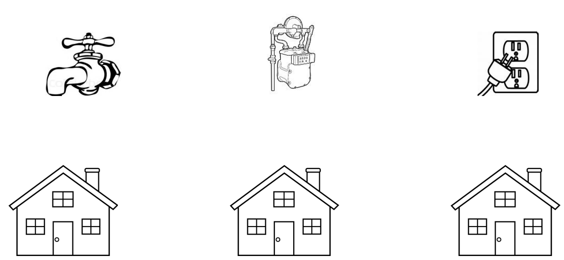
The origin of this puzzle is unknown. Sam Loyd claimed that he invented this recreational math problem about 1903. But this puzzle is MUCH older than electric lighting or even gas, Loyd most probably modified a previously existing puzzle.

Henry Dudeney , probably the greatest English puzzle designer , published the puzzle in *The Strand Magazine* in 1913.

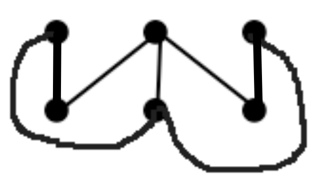
In 1917 Dudeney wrote in the book *Amusements in Mathematics* **,** “There are some half-dozen puzzles, as old as the hills, that are perpetually cropping up, and there is hardly a month in the year that does not bring inquiries as to their solution. Occasionally one of these, that one had thought was an extinct volcano, bursts into eruption. I have received an extraordinary number of letters respecting the ancient puzzle that I call "Water, Gas and Electricity". It is much older than electric lighting, or even gas, but the new dress brings it up to date. The puzzle is to lay on water, gas, and electricity, from W, G and E, to each of the three houses, A, B and C, without any pipe crossing another.



In the 1917 printing Dudeney used pictures of houses with a door and window in each house. This allowed him to suggest a solution where a supply line went from the water valve to the the first house. A second supply line form the gas meter went to the same house but went in the door and out the window and on to the second house. Using the same trick with the third house and the Gas and Electric supply lines allowed the puzzle to be solved on the flat single side of the paper. This was seen as a clever attempt at a solution but was not accepted as following the rules.

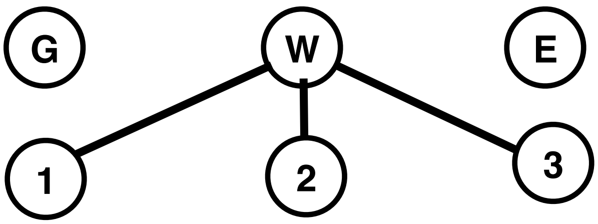


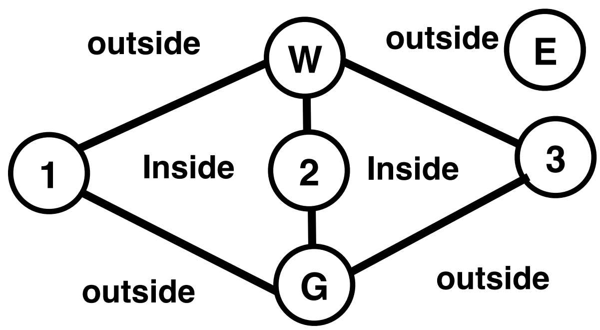
Modern mathematics requires **points** represented by dots for each of the houses and lines and curves for the paths. This requirement reduces pathway and network problems to points of no width, lines and curves.



Why is it impossible to solve this puzzle in 2 dimensions?

The diagram below left starts with 3 lines drawn from the water to each of the houses. The gas supply can be placed anywhere without changing the puzzle. It is moved below the houses so that the supply lines can be draw as straight lines. Wherever we place the gas supply the 3 lines from the gas to the 3 houses will enclose house 2. The drawings may look different and have curved paths instead of straight paths depending on the placement of the gas supply but the relationship between the supply points and supply lines will stay the same. It is clear that the 3 connections starting from the 2 utilities will enclose house 2 no mater where the water and gas supplies are placed. This forces 2 inside areas and an outside area on the plane. If the electric supply is placed outside it cannot connect to house 2. If the electric supply is placed on one of the inside areas it cannot connect to both house 1 and house 3.





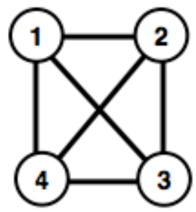
One course of study in advanced math is Topology. Topology is the study of the properties of objects that stay the same when we deform the object by bending, stretching or moving parts of the object. When we place the water supply lines we draw 3 lines. Wherever we place the gas supply the 3 lines from the gas to the 3 houses will enclose house 2. The drawings may look different and have curved paths instead of straight paths but the relationship between the supply points and supply lines will stay the same according to the theorems developed in topology. The first 2 sets of supply lines create a closed area. This is called a closed loop or closed curve in topology. We know that house 2 is inside a closed curve and the electric supply is outside the closed area.

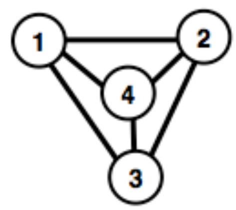
In topology a Jordan curve is a non-self-intersecting closed loop in the plane. The loop with points W 1, 3, and G form a closed loop with house 2 inside and the electric supply outside the loop. The Jordan curve theorem states that every Jordan curve divides the plane into an "interior" region bounded by the curve and an "exterior" region containing all exterior points **and any path that connects a point in the exterior region to a point in the interior region must intersect with that loop somewhere.** This means the gas line form the supply cannot be connected to house 2 without intersecting the closed loop of supply lines. This theorem can be extended to 3 objects in 3 dimensions.

There are many theorems involving pathways, networks and 2 and 3 dimensional objects that involve simple algebra equations. This fact means that students that can work with simple algebraic equation can be exposed to the beginning concepts that are explored in advanced mathematics courses.

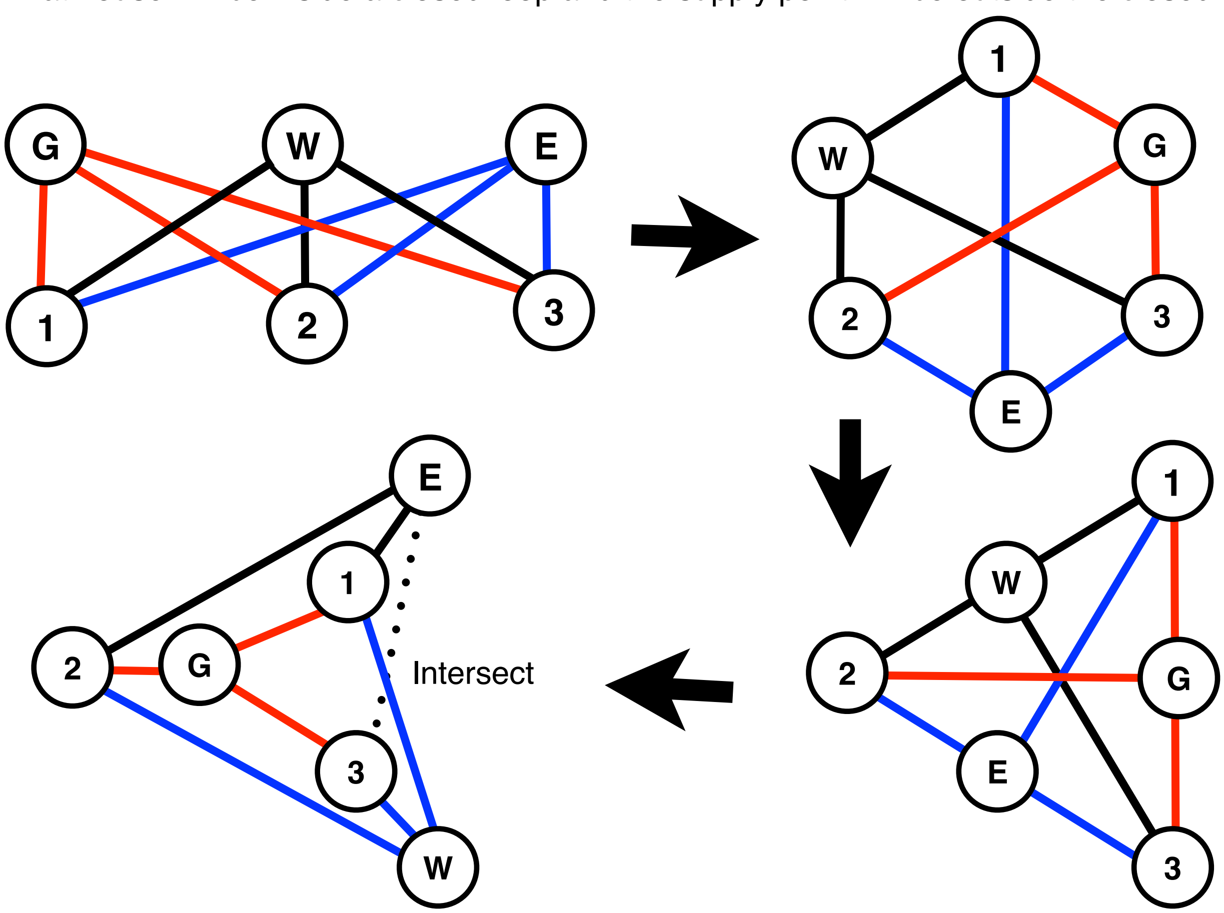
Topology and Graph Theory

The graph below left has all 4 vertices connected to each other by 6 paths (edges). Some of the paths (edges) intersect. A question in topology would ask if we can move the points and paths on the plane so that all 4 vertices are still connected to each other but using paths (edges) that do not intersect. If we slide points 3 and 4 to from the diagram on the right we see that all 4 vertices are still connected to each other by 5 paths (edges) and that there are no more edge intersections.





Graph 1 connects all 3 supply points with all 3 houses. There are 9 paths (edges) and many of the edges intersect which is not allowed. Rearranging the points to get graph 2 gives the same problem but only 3 of the paths intersect. Rearranging the points to get graph 3 still has 3 of the paths intersect. Rearranging the points to get graph 4 reduces the number of intersections to 1. If we continue to transformation the points one of the houses will have only 2 supply lines attached. That house will be inside a closed loop and the supply point will be outside the closed loop.



Note: There is no solution using the closed flat surface of the paper. This is not a proof of the statement. No matter how many examples we show unless you can show every possible configuration you cannot prove the statement by showing examples.

An algebraic proof of the fact there is no solution using Euler'd graph Formula.

A famous mathematician developed many formulas concerning closed loops. He was able too prove that these equations must be true for all closed loops.

The formula we want to look at is called **Euler'd graph Formula. V – E + F = 2**

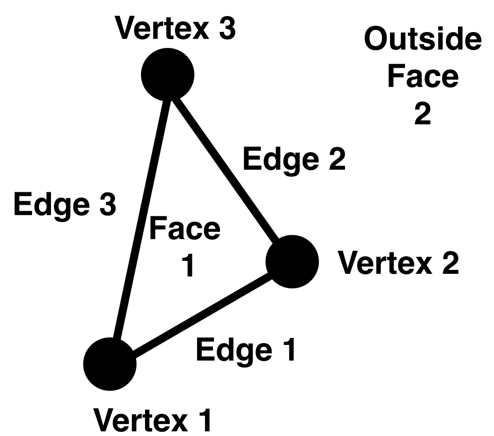
There are 3 variables in the formula.

V = The number of vertex or points.

E = The number of different paths that connect the vertex.

F = the number of closed areas plus the outside area. The outside area is a face.

The network below has 3 vertex connected by 3 paths (edges) that do not intersect. The network has 1 closed area and 1 outside area for a total of 2 faces. Since the paths do not intersect Euler'd graph Formula must be true.



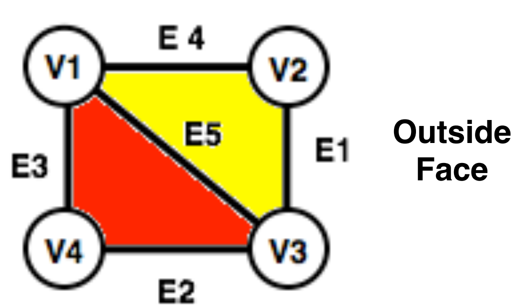
V = 3 , E = 3 , F = 2

Does V – E + F = 2

3 – 3 + 2 = 2

Yes

The network below has 4 vertex connected by 5 paths (edges) that do not intersect. The network has 2 closed area and 1 outside area for a total of 3 faces. Since the paths do not intersect Euler'd graph Formula must be true.



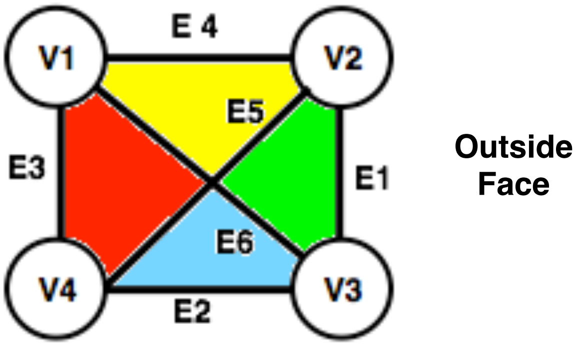
V = 4 , E = 5 , F = 3

Does V – E + F = 2

4 – 5 + 3 = 2

Yes

The network below has 4 vertex connected by 6 paths (edges) but some of the paths do intersect. Since the paths do intersect Euler'd graph Formula must **NOT** be true.
The network has 4 closed area and 1 outside area for a total of 5 faces.



V = 4 , E = 6 , F = 5

Does V – E + F = 2

4 – 6 + 5 = 7

The total is NOT 2

No

We can use Euler'd graph Formula to prove that this puzzle is unsolvable. If we can show that Euler'd graph Formula will never work for the Water, Gas and Electric network then we have proven the puzzle has no solution.

In our puzzle, the houses and utility suppliers together represent the Vertices, and the Faces are the areas inside a closed loop of Edges (this formula counts the area outside the graph as one of the Faces). Important: there can't be any Vertices in the middle of a Face.

Vertex

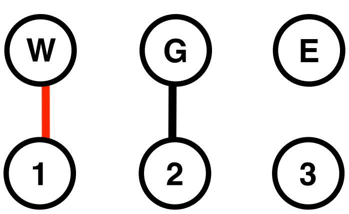
There are 3 utilities and 3 houses so there are 6 vertices. **V = 6**

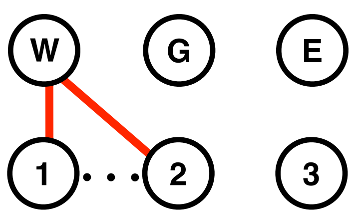
Edges

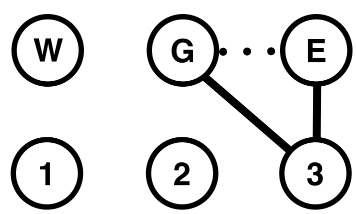
Each utility has 3 paths from it to connect the 3 houses. There are 3 utilities so it takes 3 x 3 = 9 paths to complete all the connections. This mean we have 9 edges. **E = 9**

Faces

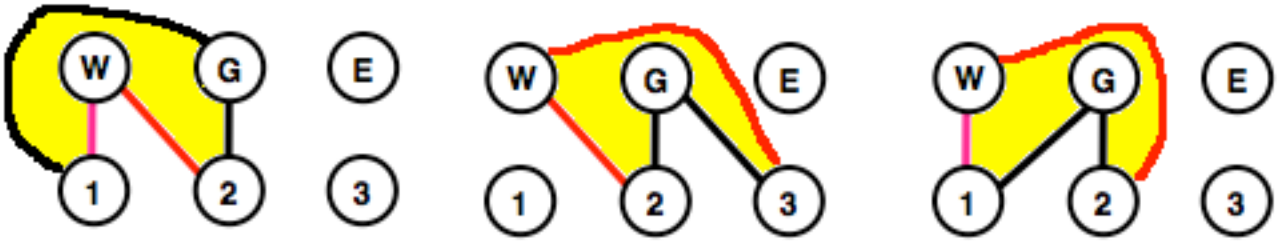
Now lest see what we can say about the number of faces that must exist. There cannot be closed area with created with 3 paths. This would be the equivalent of a triangle. The first 2 paths would have to connect **2 utilities to 2 houses** or 1 utilities to 2 houses or 2 utilities to 1 house. The third side would require a house to house or utility to utility connections which is not allowed. No house to house or utility to utility paths are allowed. This means that **it takes at least 4 edges to enclose an area.**



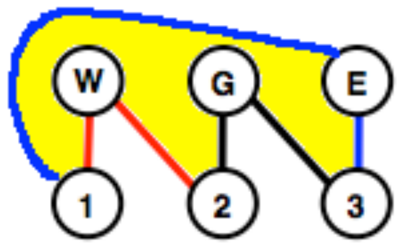




It takes at least 4 edges to enclose an area. The correct combination of 2 different utilities to 2 houses will give an enclosed area with 4 sides.



We state that **It takes at least 4 edges to enclose an area** because it can be done with more than 4 paths. The area below has 6 sides..



We know 3 things that must be true about a network with no intersecting pates.

There are **6 vertices. V = 6**

This mean we have **9 edges**. **E = 9**

Each face has **at least 4 edges.**

Each edge belongs to exactly two faces

Euler's formula states that V – E + F = 2

solving for F we get **F = 2 + E – V**

Using Euler's formula with **E = 9 V = 6 we get F = 2 + 9 – 6 so F = 5**

If the network is a solution to the problem then it must have 5 faces.

Every face has at least 4 edges, so the number of edges for the 5 faces is at least 4 x 5 = 20 edges. This counts each edge twice, as every edge is a boundary for 2 Faces. So, the smallest number of Edges is 20 / 2 = 10 edges. We know, however, that there are only 9 Edges. A figure can not have 9 edges and 10 edges at the same time.

This is a contradiction so a solution to the puzzle must be impossible.