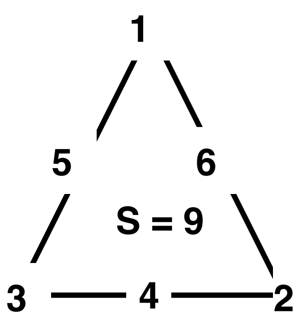
Perimeter Magic Polygons

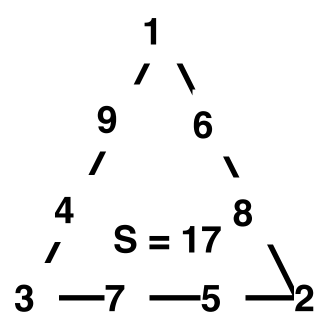
In 1972, Terrel Trotter, Jr., then a math teacher in Urbana Illinois, published an article called Magic Triangles of Order n. In 1974, he published a follow up article called Perimeter Magic Polygons. In it he extended the concepts for Magic Triangles and generalized then to include other polygonal shapes. Since that time, other writers have expanded on the same theme.

Perimeter Magic Polygons (PMP) refers to a regular polygon where all sides the same length and all angles the same size. The order n of the polygon refers to the number of integers on each side of the polygon. **Consecutive whole numbers starting with 1** are placed in the perimeter positions so the totals for all the rows is the same number.

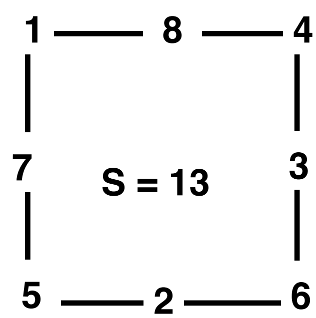
A third order
n = 3 Triangle
using 1 to 6
Magic Sum for
each side is 9



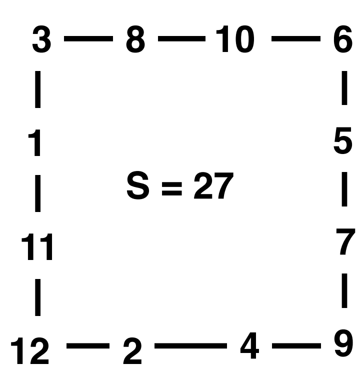
A forth order
n = 4 Triangle
using 1 to 9
Magic Sum for
each side is 17



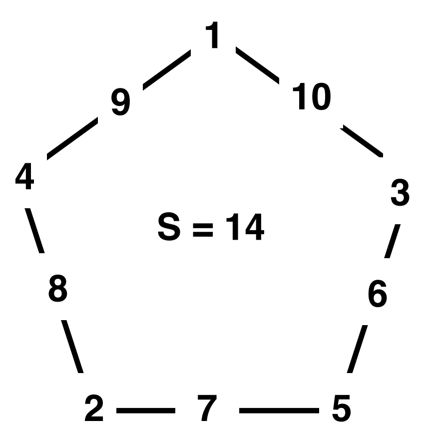
A third order
n = 3 Square
using 1 to 8
Magic Sum for
each side is 13



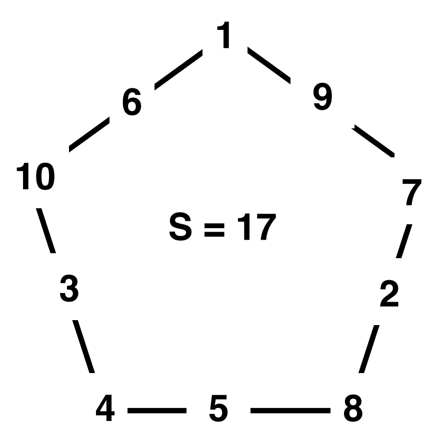
A fourth order
n = 4 Square
using 1 to 12
Magic Sum for
each side is 27



A third order
n = 3 hexagon
using 1 to 19
Magic Sum for
each side is 17



A third order
n = 4 hexagon
using 1 to 10
Magic Sum for
each side is 17



Order 3 Perimeter Magic Triangles

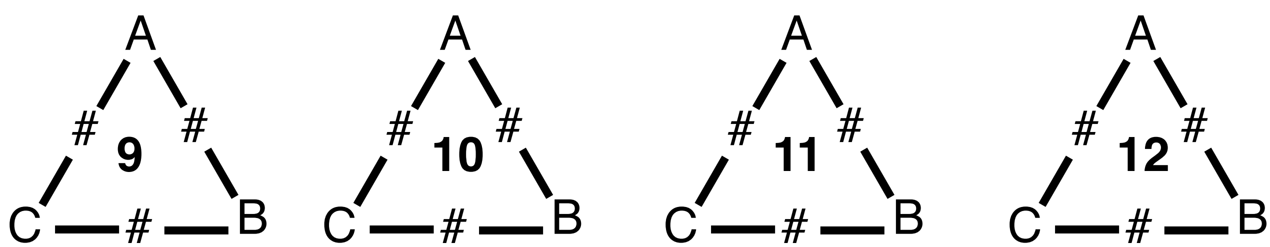
An order 1 magic triangle would be a single number so it is not considered. The numbers 1 , 2 and 3 cannot be placed to form an order 2 magic triangle so there are no order 2 magic triangles. For this reason we start with Order 3 magic triangles.

How many different possible magic sums does a triangle of order 3 have?

For magic perimeter polygons the number of different possible magic sums is S

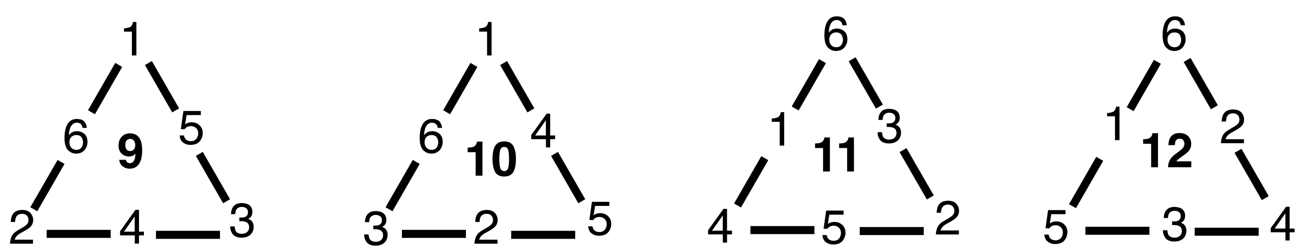
S = 3n – 5 where n is the number of integers per side.
S = 3 (3) – 5 = 4
There are only 4 possible magic sums for order 3 MP Triangles

We will look at formulas that predict the possible magic sums for each permitter magic polygon but for now we will say that the possible magic sums for order 3 triangles are 9 , 10, 11 and 12



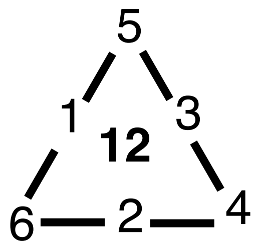
This formula does not tell you what the 4 sums are but it does tell you how many different sums there are. After you find the 4 solutions for the magic sums of 9, 10, 11 and 12 you know that no more exist.

There are only 4 possible order 3 magic triangles.



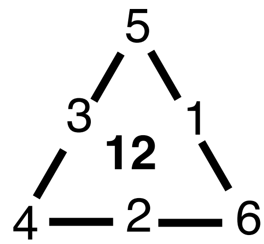
Reflections

Reflections of one solution are not considered to be unique solutions. Each basic solution for a magic triangle has one horizontal reflection. These numbers would vary for other polygons.



Magic triangle solutions.

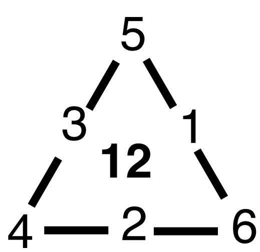
These 2 solutions are
horizontal reflections of each other.
They are considered the same solution.



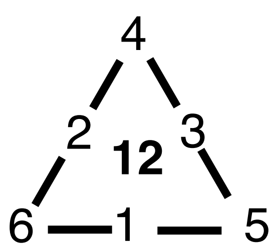
Rotations

Rotations of a solution are not considered to be unique solutions. Each basic solution for a magic triangle has has one horizontal reflection. Each of the 2 reflections has 3 rotations for a total of 6 apparently different solutions. Any one of these six could be considered the basic solution but only one of these 6 should be listed as a basic solution. For other polygons, these numbers would vary.

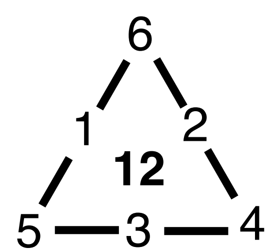
All 6 of these solutions are considered the same solution

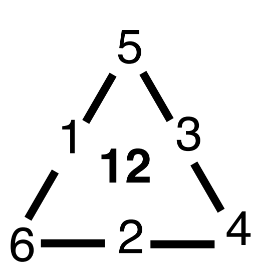


Rotate 120 degrees to get the triangle on the right.

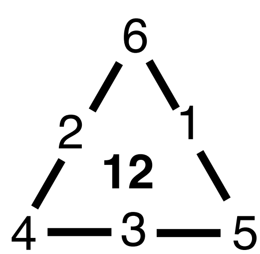


Rotate 120 degrees to get the triangle on the right.

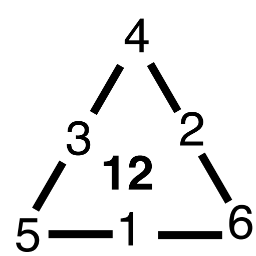




Rotate 120 degrees to get the triangle on the right.



Rotate 120 degrees to get the triangle on the right.



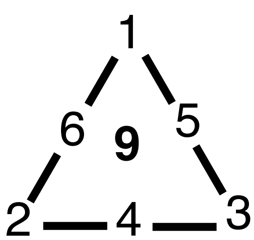
Complements

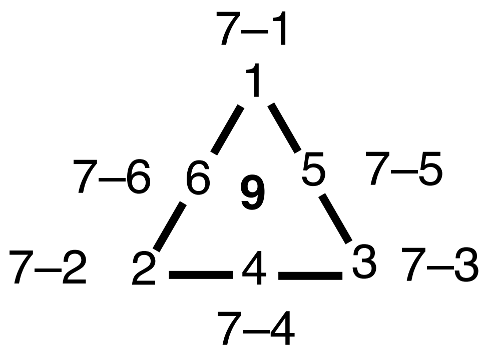
For any given solution, there is always a complement solution. It is obtained by subtracting each number in a basic solution from the sum of the first and last numbers in the series. If your puzzle uses the numbers from 1 to 6 to fill the positions on the triangle then another solution is obtained by subtracting each number in the first basic solution from 7 (ie. The sum of the first (1) and last number (6) used to fill the positions. This feature applies to magic squares, cubes etc.

If the integers used are from 1 to 6
then add the first and last numbers in the sequence to get 7

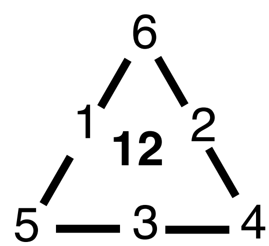
Solution A's complement solution can be found by subtracting each number in solution A from 7

Basic Solution A

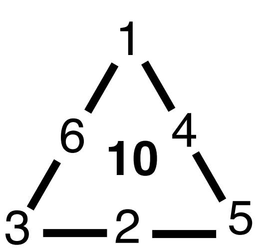


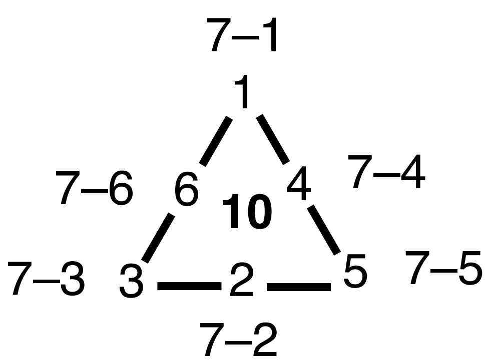


Basic Solution B

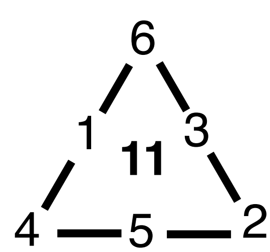


Basic Solution C





Basic Solution D



Use complements to save time finding solutions

For polygons of higher orders, with many solutions, it is necessary to only **find solutions for the smallest half of the possible sums.** The higher magic sums may then be obtained simply by finding the complements of these lower solutions.

A standard form for listing solutions

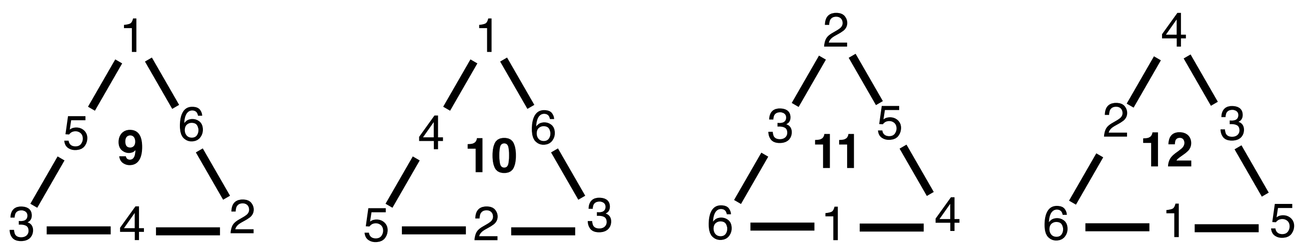
Many Perimeter Magic Polygons puzzles have an answer list with the solutions in whatever order they were found. There is no attempt to decide which of the repeated solutions will be chosen to represent that set of equivalent solutions.

There are many ways to list magic perimeter solutions. Some people list by side totals or vertex totals. For polygons with few solutions, this step is not important. However, as the number of solutions increases, an organized list is necessary in order to prevent duplicate answers. One common way to order the solutions is given below.

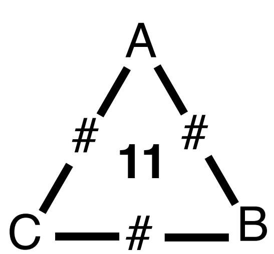
To facilitate putting polygon solutions in order:

1. Place the l**owest vertex number at the top, Letter A**
2. Then move clockwise around the polygon.
3. Enter vertex numbers in ascending (low to high) order.
4. Enter the side numbers. Where there are more then 2 side numbers use ascending order.
5. Arrange the list of solutions in ascending order of the top vertex and ascending order of the other vertex.

Order 3 Triangles solutions in list order:



Basic Solution



A, B and C are the vertex numbers.
The other 3 numbers are between the vertex.

	vertex A		vertex B		vertex C		Magic sum	Vertex sum
1.	**1**	6	**2**	4	**3**	5	**9**	**6**
2.	**1**	6	**3**	2	**5**	4	**10**	**9**
3.	**2**	5	**4**	1	**6**	3	**11**	**12**
4.	**4**	3	**5**	1	**6**	2	**12**	**15**

Order 4 Perimeter Magic Triangles

For magic perimeter polygons the number of different possible magic sums is S

S = 3n – 5 where n is the number of integers per side.
S = 3 (4) – 5 = 7

There are only 7 possible magic sums for order 3 MP Triangles

We will look at formulas that predict the possible magic sums for each permitter magic polygon but for now we will say that for order 4 triangles the **possible** magic sums are 17, 18 , 19, 20, 21 , 22 and 23 per side. We find that magic sums of 18 and 22 do not exist.

The formula **S = 3n – 5 where n is the number of integers per side** find the possible magic sums for an order n magic polygon. Another set of formulas will tell us a range for the possible sums. On very rare occasions, there may be no solution for a particular value of S. However, only two such cases are known.

They are for a 4^{th} order triangle with S = 18 or 22,
and,
3^{rd} order pentagon with S = 15 or 18.

It is unfortunate that the first exception is the second example we look at.

Basic
Solution

A
**/ **
# #
**/ **
# #
S = ?
**/ **
C — # — # — B

A, B and C are the vertex numbers.
The 2 numbers between vertex A and B are listed between A and B.
The 2 numbers between vertex B and C are listed between B and C.
The 2 numbers between vertex C and A are listed at the end

vertex A			**vertex B**			**vertex C**			**Magic sum**	**Vertex sum**
1	5	9	**2**	4	8	**3**	6	7	**17**	**6**
1	6	8	**2**	5	7	**3**	4	9	**19**	**6**
1	5	9	**4**	6	2	**7**	3	8	**19**	**12**

Examples of the first 3 solutions

1 5 9 **2** 4 8 **3** 6 7

1
**/ **
7 5
**/ **
6 9
S = 17
**/ **
3 — 8 — 4 — 2

1 6 8 **2** 5 7 **3** 4 9

1
**/ **
9 6
**/ **
4 8
S = 17
**/ **
3 — 7 — 5 — 2

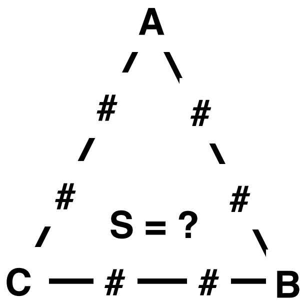
1 5 9 **4** 2 6 **7** 3 8

1
**/ **
8 5
**/ **
3 9
S = 19
**/ **
7 — 6 — 2 — 4

The 18 Basic Solutions To Order 4 Perimeter Magic Triangles

Note that there are no solutions with magic sums of 18 and 22. The number of solutions for each sum should be symmetrical.

Sum	17	19	20	21	23	
# of solutions	2	4	6	4	2	= 18



A, B and C are the vertex numbers.
The 2 numbers between vertex A and B are listed between A and B.
The 2 numbers between vertex B and C are listed between B and C.
The 2 numbers between vertex C and A are listed at the end

	vertex A	#	#	vertex B	#	#	vertex C	#	#	magic sum	vertex sum
Solution 1.	1	5	9	2	4	8	3	6	7	17	6
Solution 2.	1	5	9	4	2	6	7	3	8	19	12
Solution 3.	1	6	8	2	5	7	3	4	9	17	6
Solution 4.	1	6	8	4	3	5	7	2	9	19	12
Solution 5.	1	6	8	5		2	4	9	7	3	20
Solution 6.	2	4	9	5	1	6	8	3	7	20	15
Solution 7.	2	5	9	3	1	8	7	4	6	19	12
Solution 8.	2	6	7	5	3	4	8	1	9	20	15
Solution 9.	2	6	8	3	4	5	7	1	9	19	12
Solution 10.	3	2	9	7	1	5	8	4	6	21	18
Solution 11.	3	4	8	5	2	6	7	1	9	20	15
Solution 12.	3	4	8	6	1	5	9	2	7	21	18
Solution 13.	3	5	6	7	2	4	8	1	9	21	18
Solution 14.	3	5	7	6	2	4	9	1	8	21	18
Solution 15.	4	2	9	5	1	8	6	3	7	20	15
Solution 16.	4	3	8	5	2	7	6	1	9	20	15
Solution 17.	7	2	6	8	1	5	9	3	4	23	24
Solution 18.	7	3	5	8	2	4	9	1	6	23	24

A formula for finding the number of possible magic sums for a magic polygon.

The number of possible sums for a polygon of order n is **S = 3*n* – 5.**

A formula for finding the range of magic sums for any magic permitter polygon

k = the number of sides of the polygon **n** = the number of integers per side (the order)

There are two formulas for finding the range of sums, depending on the values of n and k.

If $[n \text{ is even }]$ or $[n \text{ AND } k \text{ are BOTH ODD }]$ then

$$\text{Minimum Magic Sum } = \frac{k \bullet n^2 - 2k \bullet n + 2k + n}{2} \quad \text{and Maximum Magic Sum } = \frac{kn^2 - 2k + n}{2}$$

If $[n \text{ is ODD AND } k \text{ EVEN }]$ then

$$\text{Minimum Magic Sum } = \frac{kn^2 - 2kn + 2k + n + 1}{2} \quad \text{and Maximum Magic Sum S } = \frac{k \bullet n^2 - 2k + n - 1}{2}$$

These formulas indicate the range of possible magic sums that **may be found**. On 2 occasions, there is no solution for a particular value of S. They are for a **4th order triangle with S = 18 or 22**, and a **3rd order pentagon with S = 15 or 18.**

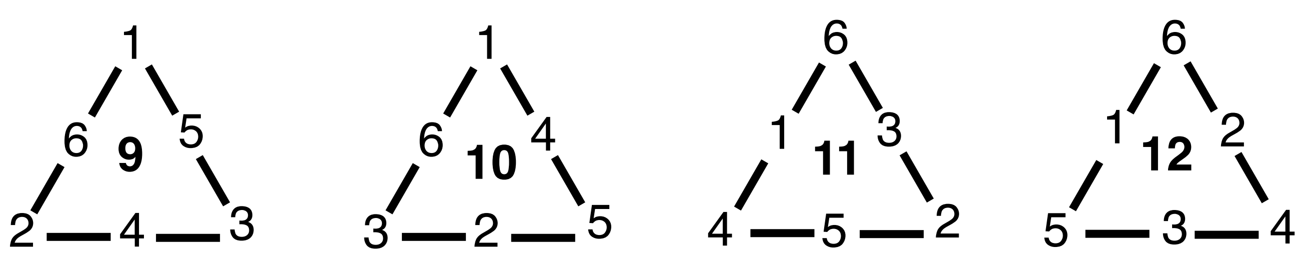
Perimeter Magic Triangles

Here is summary information on orders 3 to 8 perimeter magic triangles.

Triangle of Order	**3**	**4**	**5**	**6**	**7**	**8**
Minimum S	9	17	28	42	59	79
Maximum S	12	23 *	37	64	74	97
Integers used	1–6	1–9	1–12	1–15	1–18	1–21
Number of basic solutions	4	18	1356?	?	?	?

Note: * For order 4 triangles there are no solutions with sums of 18 or 22.

There are only 4 possible order 3 magic triangles.

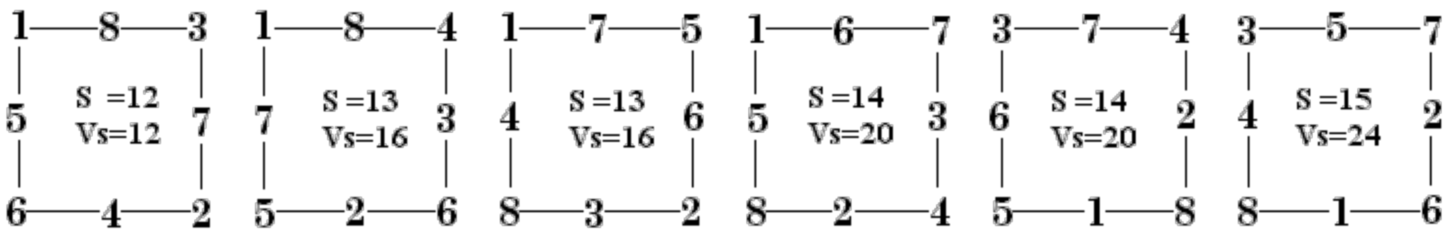


Perimeter Magic Squares

This table shows relevant information for 4 sided PMPs.

Squares of Order	3	4	5	6	7	8
Minimum S	12	22	37	55	78	104
Maximum S	15	30	48	71	97	128
Integers used	8	12	16	20	24	28
Number of basic solutions	6	146?	?	?	?	?

These are the 6 basic solutions for an order 3 square. Any other solutions are rotations or reflections of these 6

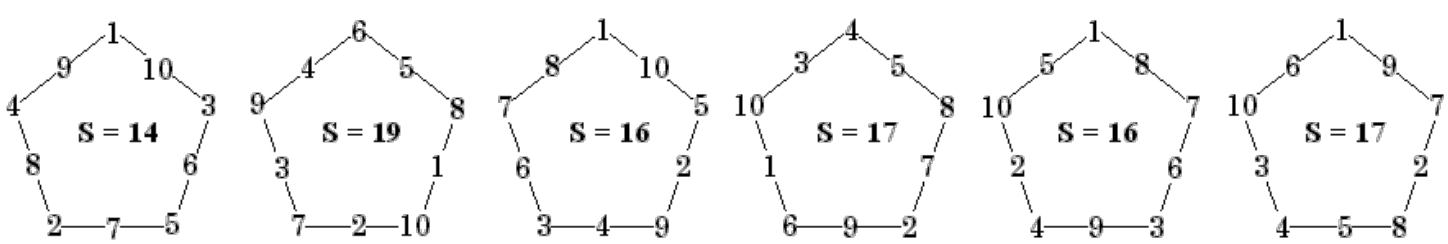


Perimeter Magic Pentagons

This table shows relevant information for 4 sided PMPs.

Squares of Order	3	4	5	6	7	8
Minimum S	14	27	45	68	96	129
Maximum S	19	37	60	88	121	159
Integers used	10	15	20	25	30	35
Number of basic solutions	6	6074?	?	?	?	?

There are the 6 basic solutions for an order 3 pentagon . Any other solutions are rotations or reflections of these 6

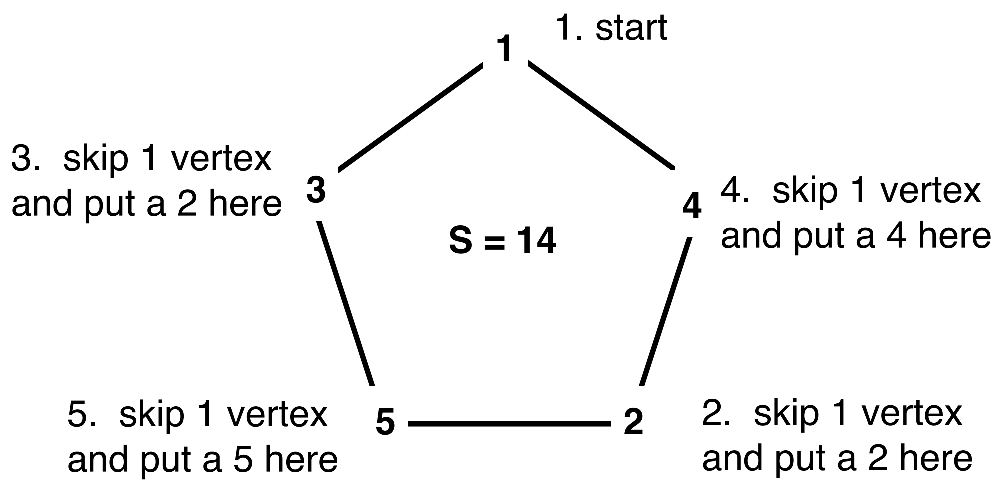


A Method for Constructing Odd Order PMPs

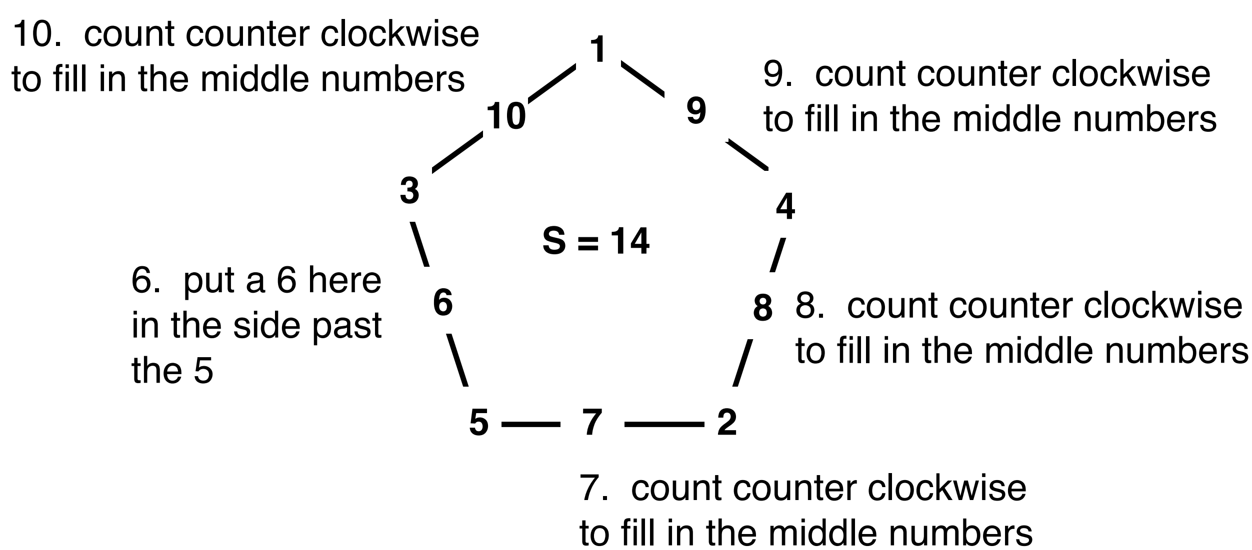
Start at the top and number every other vertex until your next move is blocked
Place the next number in the middle of the side past where you are at.
Count counter clockwise in order, placing the numbers in the middle locations between the vertex.

Pentagon Example

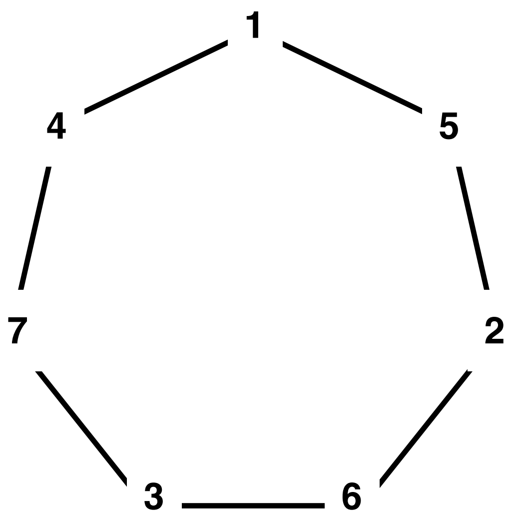
Start at the top with a 1.
Move clockwise skipping a vertex each time and numbering the next vertex.
Do this until the next step will land on a vertex that already has a number on it.
In this case you stop at 5.

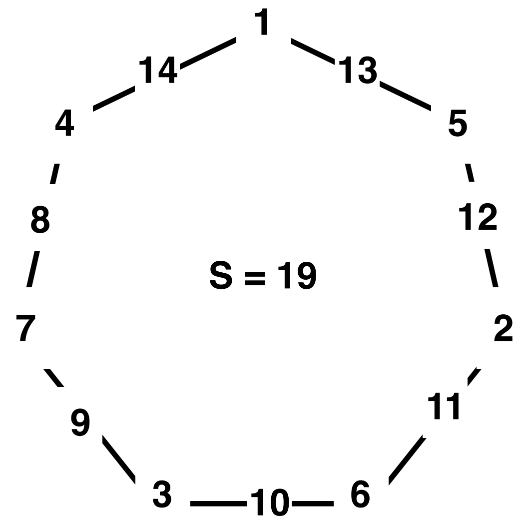


Place the next number (6) in on the side that is past 5,
Move counter clockwise numbering the next space between the vertex in order.

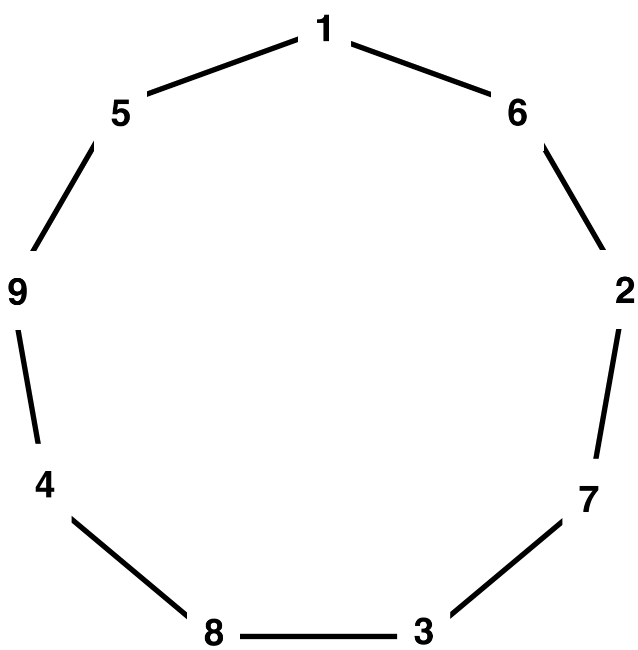


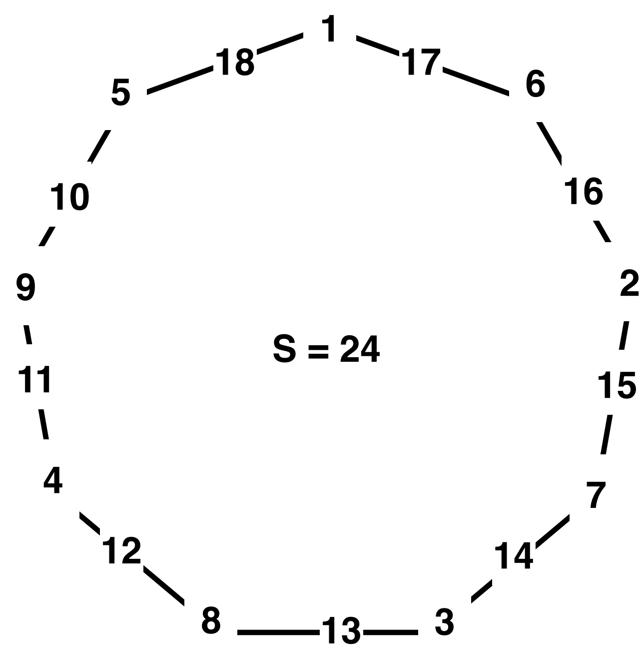
7 sided Polygon Example





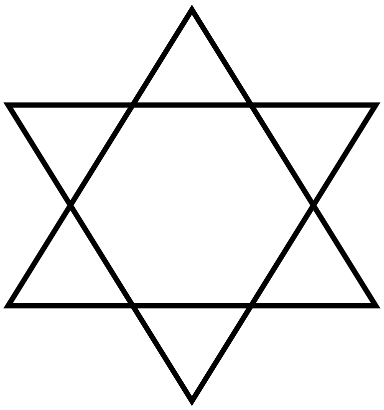
9 sided Polygon Example

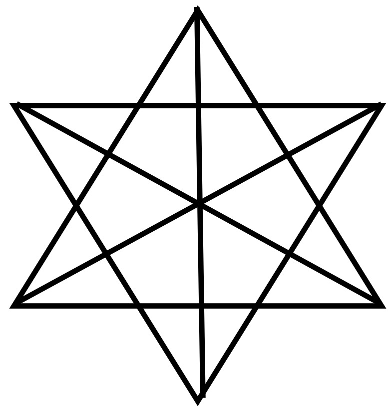




Magic Stars

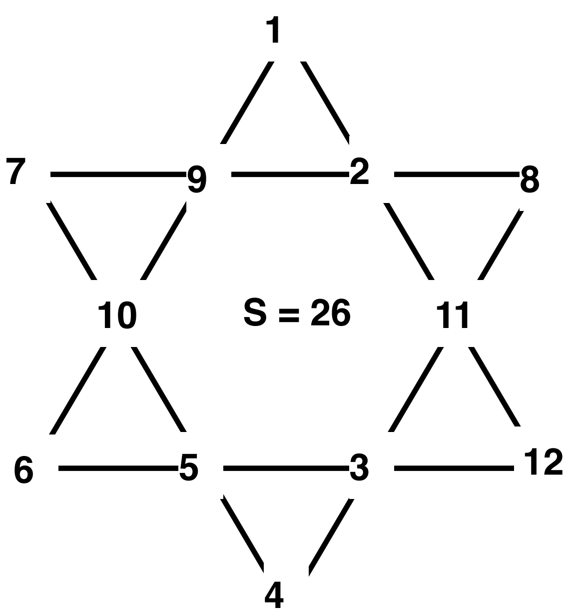
Magic Stars are normally considered to be 5 , 6, 7 ... pointed stars that are drawn with lines connecting some or all of the outer vertex points of the star. **The 6 pointed star on the left below** is the star most commonly used with students. It does not connect all of the outer vertex points like the star on the right.

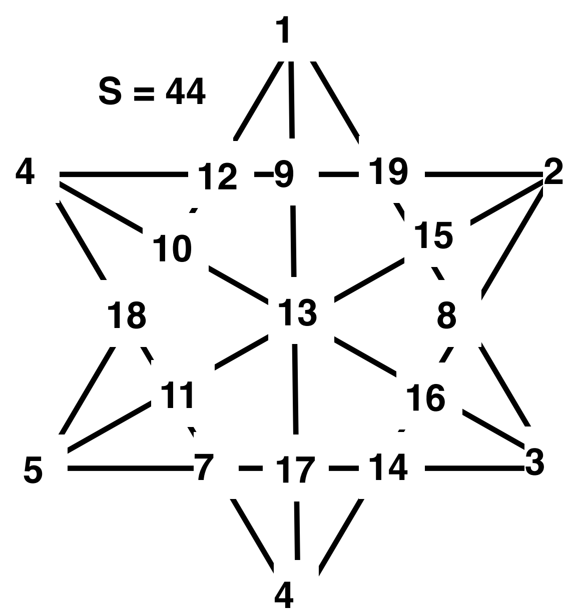




The 6 pointed star on the left below has the numbers 1 to 12 placed at the intersections of the line segments joining the vertex in such a way that the 4 numbers on each of the 6 segments add to 26

The 6 pointed star on the right below has the numbers 1 to 18 placed at the intersections of the line segments joining all the vertex in such a way that the 5 numbers on each of the 9 segments add to 44





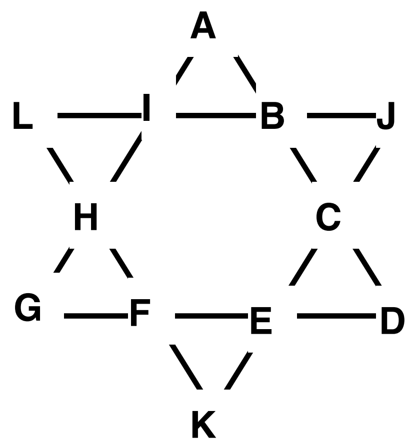
It can be proven that no star polygons with fewer than 5 points exist, and the construction of a normal (uses the number 1 to n) 5-pointed magic star turns out to be impossible. The smallest examples of normal magic stars are therefore 6-pointed stars.

Order-6 Magic Star Characteristics

1. The magic sum(S) for each line is 26 **for all solutions**
2. Order-6 is the lowest order magic star possible using the consecutive numbers 1 to n.
4. It is the only order star where there are solutions that have all the vertex points summing to the magic sum. There are six such solutions,.

5. The Order-6 star is the only magic star that does not have at least one continuous pattern.

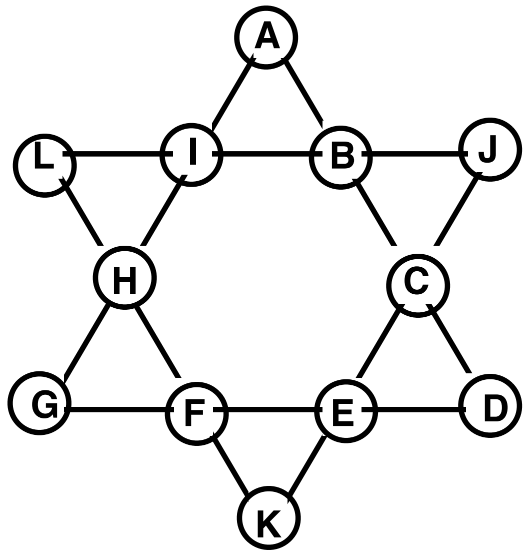
The 20 Basic solutions for a 6 point Magic Star



A	B	C	D	E	F	G	H	I	J	K	L
1	2	11	12	3	5	6	10	9	8	4	7
1	2	11	12	4	3	7	8	10	5	6	9
1	2	12	11	3	4	8	7	10	5	6	9
1	2	12	11	4	5	6	10	9	7	3	8
1	3	10	12	2	4	8	6	11	5	9	7
1	3	10	12	2	7	5	9	11	8	6	4
1	3	12	10	4	7	5	11	9	8	2	6
1	4	10	11	5	3	7	6	12	2	9	8
1	4	11	10	2	9	5	8	12	7	6	3
1	4	12	9	5	2	10	7	8	3	6	11
1	5	8	12	3	2	9	6	10	4	11	7
1	5	11	9	3	2	12	6	7	4	8	10
1	5	11	9	3	8	6	12	7	10	2	4
1	5	12	8	2	6	10	4	11	3	9	7
1	5	12	8	7	2	9	10	6	4	3	11
1	6	11	8	2	7	9	4	12	3	10	5
1	6	12	7	3	5	11	4	10	2	9	8
1	7	8	10	2	3	11	5	9	4	12	6
1	7	8	10	4	3	9	5	11	2	12	6
1	7	10	8	3	6	9	4	12	2	11	5

Any solution in the basic set of 20 cannot be rotated or reflected to find one of the other solutions. Each of the 20 basic solutions in this set can be transformed into 3 other solutions by switching given vertex numbers. This produces 80 total solutions.

Basic Solution A

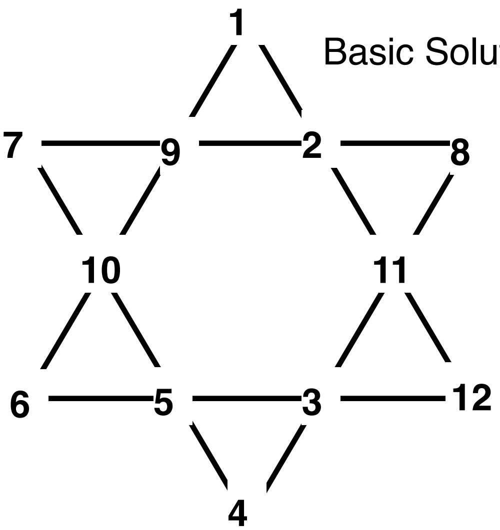


Start with Solution A
Switch
L and F
I and G
J and E
B and D
to get New Solution A2

Start with Solution A
Switch
A and C
I and J
G and E
H and K
to get New Solution A3

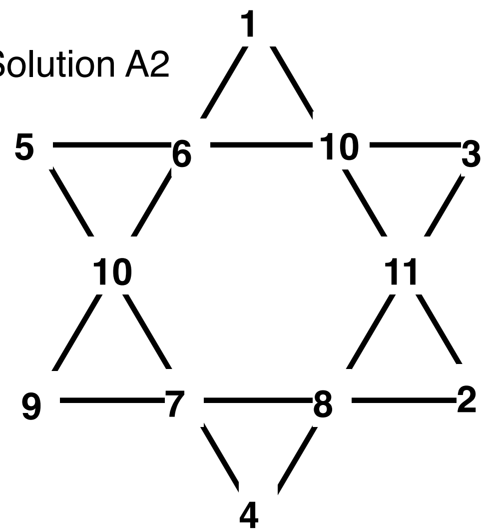
Start with Solution A
Switch
A and H
L and B
F and D
C and K
to get New Solution A4

Basic Solution A

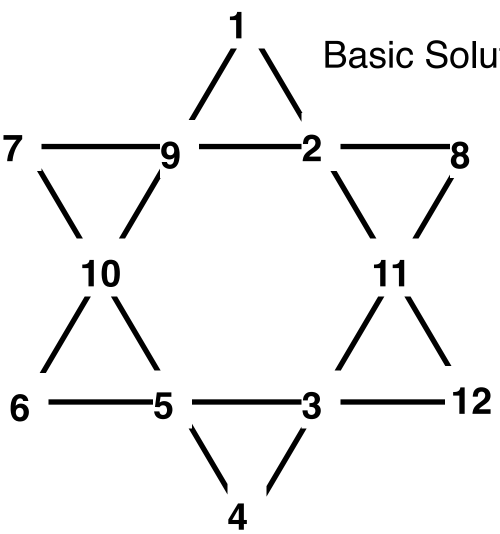


Start with A
Switch the
9 and 6
7 and 5
2 and 12
8 and 3
to get A2

New Solution A2

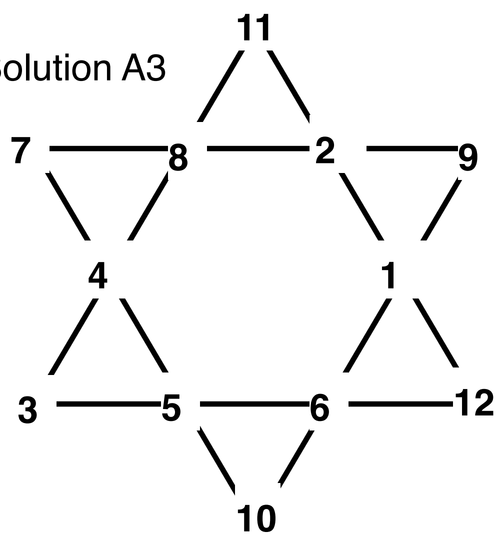


Basic Solution A

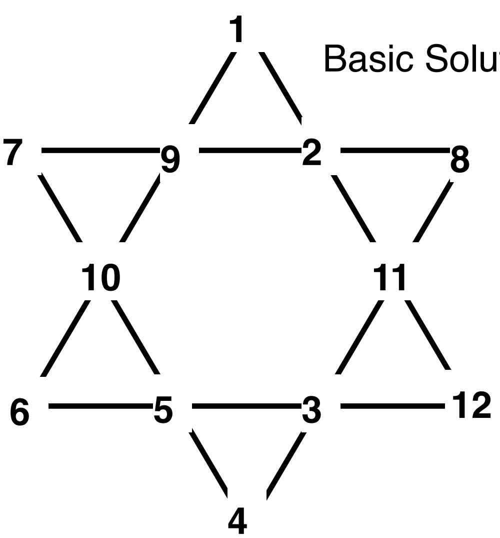


Start with A
Switch the
9 and 8
1 and 11
6 and 3
10 and 4
to get A3

New Solution A3

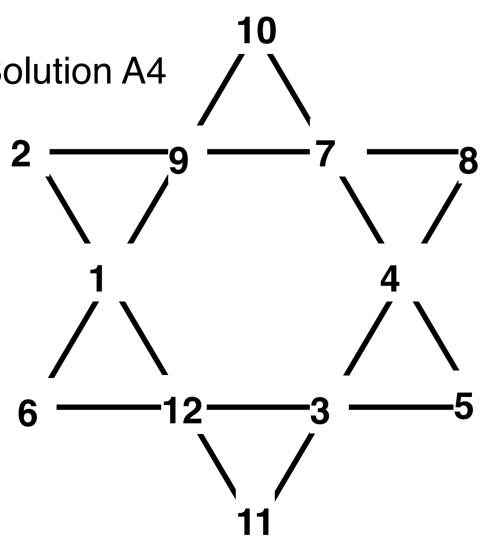


Basic Solution A



Start with A
Switch the
1 and 10
7 and 2
4 and 11
5 and 12
to get A4

New Solution A4

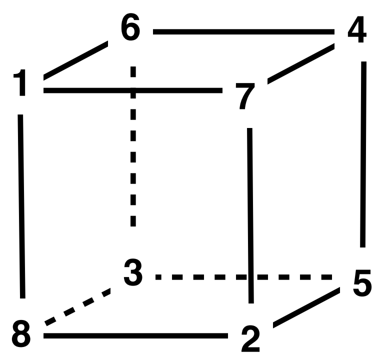


3 Dimensional Face Magic Polyhedra

The face of any polyhedra has several vertices. If the polyhedra has n vertices you can find ways to place the numbers from 1 to n in the vertex positions so that the sum of the numbers on the faces of each polyhedra all add up to the same number. We call the Face Magic Polyhedra. In most cases only Cubes and Tetrahedron are used with students.

Face Magic Cubes of Order 2

The numbers 1 to 8 have been placed on the 8 vertices of the cube below. There are 2 numbers on each edge of the cube so the cube is order 2. The 4 numbers on every face of the cube add up to 18.

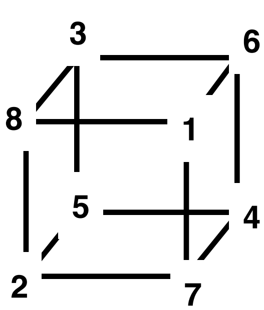


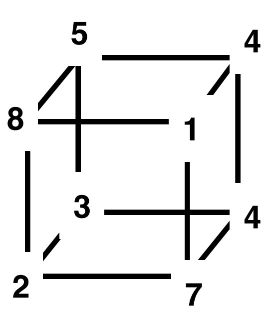
S = 18

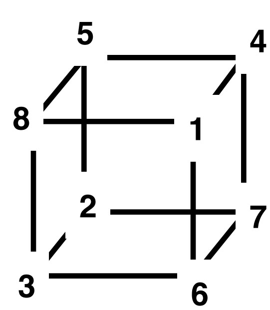
The sum of the numbers on each of the 6 faces of the cube add to 18

Top face 1 + 6 + 4 + 7 = 18	Bottom Face 8 + 3 + 5 + 2 = 18
Front face 1 + 7 + 2 + 8 = 18	Back Face 6 + 4 + 5 + 3 = 18
Left face 1 + 6 + 3 + 8 = 18	Right Face 7 + 4 + 5 + 2 = 18

The set of 1, 4 , 6, 7 must be placed on one face and the set 2 , 3 ,5 , 8 must be placed on the opposite face These two sets can be placed on opposite faces of the cube to produce the three basic solutions. All have face sums of 18. There are 3 different basic solutions based the these 2 pairs. There are another 3 solutions based on rotations.

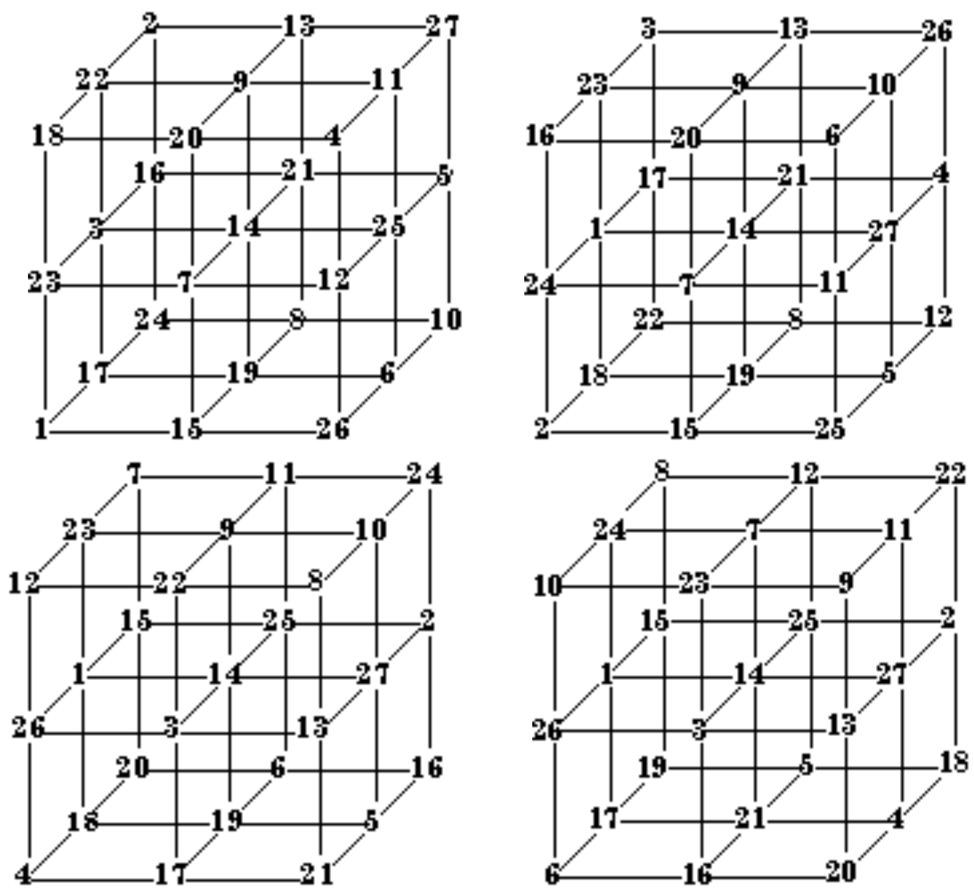






4 Basic Order 3 Magic Cubes

It was proven in 1972 that there are four basic magic cubes of order 3. These are called basic cubes because no one of them may be transformed to another one by rotations and reflections. Each of the 4 basic solutions may be shown in 47 other variations using rotations or reflections. Any Magic Cube larger than order 2 is too complex for classroom.



Edge Magic Tetrahedrons of Order 3

There are no order-2 edge magic tetrahedrons. There are no order-3 edge magic tetrahedrons using consecutive integers from 1 to 10 so there are no normal tetrahedrons we can construct with students. Orders larger than 3 are too complex for classroom use.