Coin Logic Problems

Fake Coin

1. You have 12 coins, one of which is fake. The fake coin is indistinguishable from the rest except that it is either heavier or lighter, but you don't know which. Can you determine which is the fake coin **and** whether it is lighter or heavier using a balance scale and only 3 weighings?

Twelve identical-looking coins are given, and we are told that one of them has a weight different from the other 11. The problem is to determine which coin it is and whether it is heavier or lighter, in only three weighings of these coins on a balance scale.

Note first that even to describe a solution after one has found it seems to require branching. The outcome of a weighing is that the left tray of the balance goes down or up or remains level. We encode these outcomes by 1, - 1, and 0 respectively. Each weighing involves picking a pair of subsets *L* and *R* of the same cardinality and putting them on the left and right trays, respectively. The "flow diagram" for a weighing procedure looks like this:

where for each *L* and *R* one would have to list the elements of the appropriate subsets. Compare this to the following nonbranching instructions. Let the coins be labeled *A*, *B*, ..., *L*. Then do

One easily verifies that this works. Namely, if *A* is the phony coin and it is heavy, then the outcome will be 1, 1, - 1, and if it is light, the outcome will be - 1, - 1, 1. If *B* is the culprit and is heavy, then the outcome will be 1, 0, 0, and if it is light - but wait a minute. We do not have to go through all this. All we have to do is notice that no two coins have the same or opposite "itineraries"; that is, no two coins are always on the same tray or always on opposite trays. Therefore, for each of the 24 possible states (that is, which coin is counterfeit and whether it is heavy or light), there will be a different outcome. So given the Outcome, we will know the state. For example, if we know that the outcome is - 1, 0, 1, then the coin *G* must be heavy. Note, by the way, that it makes no difference in which order the weighings are performed. Also note that we can solve a slightly harder problem in which we allow the possibility that none of the coins is counterfeit, which win be true if and only if the outcome is 0, 0, 0.

But the question now is what sort of ingenuity was required to find these three weighings. The answer-none. We let the solution give itself. The real message we wish to impart then is that old, complicated, clever solution was a waste of effort, an incorrect attitude! The new solution reasons backward from the 27 outcomes to the 24 counterfeit possibilities. Here's how it goes.

First, we make a list of 12 different outcome vectors, such that no outcome and its negative are on the list. A simple way to do this is to list the lexicographically positive vectors in lexicographic order, as in the columns of the following table.

A
B
C
D
E
F
G
H
I
J
K
L

0
0
0
0
1
1
1
1
1

1
1
1
0
1
1
1
-1
-1
-1
0
0
0
1
1
1
-1
0
1
-1
0
1
-1
0
1
-1
0

Now, for the procedure to work, we must have the same number of Is and - Is in each row. The bottom row is correct as it stands. Reversing the sign of column *C* fixes the middle row, and reversing columns F, H, J, and L takes care of the top row. So we have

A

B

C

D

E

F

G

H

I

J

K

L

0

0

0

0

1

-1

1

-1

1

-1

1

1

0

1

-1

1

-1

1

-1

0

0

0

1

-1

1

-1

0

1

-1

0

1

-1

0

1

-1

0

Now each row of the table corresponds to a weighing, namely, put the + Is on the left and - Is on the right-and there is it. Perform the weighings, write down the outcome, and read off the guilty coin from the table. (The capital letters of the tables differ by a permutation from the ones given earlier, but this clearly makes no difference.)

In some cases, nonbranching solutions are found easily. I have picked a number between I and 8 and you must guess it by asking three yes-or-no questions. Of course, everyone uses the branching, or "interactive," strategy of successive bisecting, but one need not do this. Why not just ask in advance these three questions, in any order. Is the number in the set {11, 2, 3, 4}? in the set {11, 2, 5, 6}? in the set {11, 3, 5, 7}? Of course, this would not work if the allowable question had to be of the form Is the number greater than *x*? The example shows what is going on generally. There is a set of possible states, and one wants to learn the true state. Every question, or weighing, or "experiment" gives a partition of this set. One defines the intersection of *k* partitions in the obvious way as the partition formed by all intersections of sets of the *k* partitions. Then the true state can be learned in n experiments without branching if and only if one can find n partitions whose intersection is the partition by singletons.

First Weighing	Second Weighing	Third Weighing
1234 balance 5678 the false coin is one of 9 10 11 12	1 and 2 balance 9 and 10	1 =11 12 < 11 12 is false and

First Weighing	Second Weighing	Third Weighing
1234 is heavier than 5678 the false coin is one of 9 10 11 12	1 and 9 balance 11 and 11	

First Weighing	Second Weighing	Third Weighing
1234 is lighter than 5678 the false coin is one of 9 10 11 12	1 and 9 balance 11 and 11	

Number the coins from 1 to 12. For the first weighing let us put on the left pan pills 1,2,3,4 and on the right pan pills 5,6,7,8.

There are two possibilities. Either they balance, or they don't. If they balance, then the false coin is in the group 9,10,11,12. So for our second weighing we would put 1,2 in the left pan and 9,10 on the right. If these balance then the false coin is either 11 or 12.

Weigh coin 1 against 11. If they balance, the false coin is number 12. If number 12 coin is on the heavy end it is the false coin and is heaver than the rest. If number 12 coin is on the light end it is false coin and is lighter than the rest. If they do not balance, then 11 is the false coin.

If 1,2 vs 9,10 do not balance, then the good pill is either 9 or 10. Again, weigh 1 against 9. If they balance, the good pill is number 10, otherwise it is number 9.

That was the easy part.

What if the first weighing 1,2,3,4 vs 5,6,7,8 does not balance? Then any one of these pills could be the safe pill. Now, in order to proceed, we must keep track of which side is heavy for each of the following weighings.

Suppose that 5,6,7,8 is the heavy side. We now weigh 1,5,6 against 2,7,8. If they balance, then the good pill is either 3 or 4. Weigh 4 against 9, a known bad pill. If they balance then the good pill is 3, otherwise it is 4.

Now, if 1,5,6 vs 2,7,8 does not balance, and 2,7,8 is the heavy side, then either 7 or 8 is a good, heavy pill, or 1 is a good, light pill.

For the third weighing, weigh 7 against 8. Whichever side is heavy is the good pill. If they balance, then 1 is the good pill. Should the weighing of 1,5, 6 vs 2,7,8 show 1,5,6 to be the heavy side, then either 5 or 6 is a good heavy pill or 2 is a light good pill. Weigh 5 against 6. The heavier one is the good pill. If they balance, then 2 is a good light pill.